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Math 132 Differential Topology

§ Transversality

Earlier, we saw that transversality is stable under small perturbations.

Today, we'll use Sard's theorem to show that transversal maps are <u>generic</u> in the sense that any smooth map can be deformed by an arbitrarily small amount into a transversal map.

(think as a smooth family of maps $f_s : M \to N$, $s \in S$)

<u>Thm</u> (transversality thm)

Suppose $F: M \times S \to N$ is a smooth map of manifolds, where only M has ∂, and let $P \subset N$ be a submanifold.

If both F and $F|_{\partial M \times S}$ are transversal to P, then for almost every $s \in S$, both f_s and $f_s|_{\partial M}$ are transversal to P.

proof) The preimage $Q = F^{-1}(P) \subset M \times S$ is a submanifold with boundary $\partial Q = Q \cap (\partial M \times S)$. Let $\pi : Q \to S$ be the projection.

We'll show that ① $f_s \pitchfork P$ whenever s is a regular value for π,

and that ② $f_s|_{\partial M} \pitchfork P$ whenever s is a regular value for $\pi|_{\partial Q}$.

The theorem will then follow from Sard.

Note that the second claim ② is a special instance of the first claim ①, for the case of the map $F|_{\partial M \times S} : \partial M \times S \to N$ of boundaryless manifolds, so it suffices to show ①.

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In order to show $f_s \pitchfork P$, we need to show that, for any $x \in M$ with $f_s(x) = y \in P$, the composition $T_xM \xrightarrow{d(f_s)_x} T_yN \xrightarrow{p} T_yN/T_yP$ is surjective.

$d(f_s)_x(v) = dF_{(x,s)}(v,0)$

Since $F \pitchfork P$, we know that

$$T_{(x,s)}(M \times S) = T_xM \times T_sS \xrightarrow{dF_{(x,s)}} T_yN \overset{p}{\twoheadrightarrow} T_yN/T_yP$$

is surjective. Moreover, by the regularity assumption,

$$d\pi_{(x,s)} : T_{(x,s)}Q \longrightarrow T_sS \text{ is surjective.}$$

Since $T_{(x,s)}Q \subset T_xM \times T_sS$ is in the kernel of $p \circ dF_{(x,s)}$, it follows that the restriction $p \circ dF_{(x,s)}\big|_{T_xM \times 0} = p \circ d(f_s)_x$ is surjective, as desired. ■

<u>Cor</u> Transversal maps are <u>generic</u> when the target manifold N is $\mathbb{R}^\ell$.

proof) If $f : M \to \mathbb{R}^\ell$ is any smooth map, take S to be an open ball of $\mathbb{R}^\ell$, and define $F : M \times S \to \mathbb{R}^\ell$

$$(x, s) \mapsto f(x) + s.$$

For any fixed x, this is obviously a submersion, so transversal to any submanifold $P \subset \mathbb{R}^\ell$.

Thanks to the transversality theorem, $f_s(x) = f(x) + s$ is transversal to P, for almost every $s \in S$, where s may be taken to be arbitrarily small.

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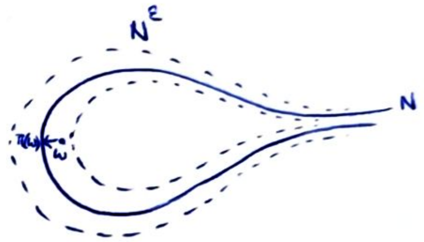
For an arbitrary target manifold $N \subset \mathbb{R}^\ell$, the proof of genericity of transversal maps is essentially the same; we just need to project the shifted map down to N.

<u>Lemma</u> (ε-neighborhood thm)

For any manifold $N \subset \mathbb{R}^\ell$, there is a smooth positive function $\varepsilon : N \to \mathbb{R}_{>0}$ such that each point $w \in N^\varepsilon := \{w \in \mathbb{R}^\ell \mid \|w - y\| < \varepsilon(y) \text{ for some } y \in N\}$ has a unique closest point $\pi(w)$ in N.

Moreover, the map $\pi : N^\varepsilon \to N$ is a submersion.



(See pp. 71-72 of [Guillemin-Pollack] for the proof.)

<u>Cor</u> Let $f : M \to N$ be a smooth map, where only M has boundary.

Then there is an open ball S in some Euclidean space and a smooth map $F : M \times S \to N$ such that $F(x,0) = f(x)$, and for any fixed $x \in M$, the map $S \to N$, $s \mapsto F(x,s)$ is a submersion.

In particular, both F and $F|_{\partial M \times S}$ are submersions.

proof) Let S be the unit open ball in $\mathbb{R}^\ell \supset N$, and define

$$F(x,s) = \pi\big(f(x) + \varepsilon(f(x))s\big).$$

Since $\pi : N^\varepsilon \to N$ restricts to the identity on N, $F(x,0) = f(x)$.

Since $s \mapsto F(x,s)$ is a composition of two submersions, $s \mapsto f(x) + \varepsilon(f(x))s$ and π, it is a submersion ■

That transversality is generic follows directly:

<u>Thm</u> (transversality homotopy theorem)

For any smooth map $f: M \to N$ and a submanifold $P \subset N$, where only M has ∂, there exists a smooth map $g: M \to N$ homotopic to f such that $g \pitchfork P$ and $g|_{\partial M} \pitchfork P$.

proof) For the family of maps F in the corollary, the transversality thm implies that $f_s \pitchfork P$ and $f_s|_{\partial M} \pitchfork P$ for almost all $s \in S$, but each f_s is homotopic to f, with homotopy $\begin{aligned} M \times I &\longrightarrow N \\ (x,t) &\longmapsto F(x,ts) \end{aligned}$. ∎

We'll actually need a slightly stronger form of this theorem:

<u>Thm</u> (~~extension~~ relative transversality homotopy theorem)

In the setting as above, where P is a <u>closed</u> submanifold, suppose $C \subset M$ is a closed subset such that $f \pitchfork P$ on C and $f|_{\partial M} \pitchfork P$ on $C \cap \partial M$.

Then there exists a smooth map $g: M \to N$ homotopic to f such that $g \pitchfork P$, $g|_{\partial M} \pitchfork P$, and $g = f$ on a neighborhood of C.

(See pp. 72-73 of [Guillemin-Pollack] for the proof. The idea is to modify $F: M \times S \to N$ by setting $\begin{aligned} G: M \times S &\to N \\ (x,s) &\mapsto F(x, \tau(x)s) \end{aligned}$ where $\tau: M \to [0,1]$ is a smooth bump function that is 0 on C and 1 outside of an open neighborhood.)

<u>Cor</u> If $f: M \to N$ is a smooth map such that $f|_{\partial M}: \partial M \to N$ is transversal to $P \subset N$, then there exists a map $g: M \to N$ homotopic to f rel ∂ (i.e. $g|_{\partial M} = f|_{\partial M}$) such that $g \pitchfork P$.